8. (a) State and prove integration by parts formula for the Riemann integral. 10

(b) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Assume that there exists a function $F : [a, b] \to \mathbb{R}$ such that $F' = f$ on $[a, b]$. Prove that

$$\int_a^b f(t)dt = F(b) - F(a).$$ 10

Exam. Code : 211001
Subject Code : 4952

M.Sc. Mathematics 1st Semester (Batch 2021-23)

REAL ANALYSIS–I

Paper—MATH-551

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **FIVE** questions in all, selecting at least **ONE** question from each section. The fifth question may be attempted from any section. All questions carry equal marks.

SECTION—I

1. (a) Let $\phi \neq A \subset \mathbb{R}$ and Ω be the set of sequences with terms from A. Prove that Ω is either a singleton set or uncountable. 8

(b) Prove that the set of algebraic numbers is countable. 7

(c) Prove that every compact subset of a metric space is bounded. Is the converse true ? 5

2. (a) Prove that every k-cell is compact. 10

(b) Prove that the set of limit points of a set is always closed. Is the same true for the set of adherent points ? 10

SECTION—II

3. (a) Let $k \in \mathbb{N}$ and $\phi \neq X \subset \mathbb{R}^k$. Prove that X is a complete subspace of $\mathbb{R}^k$ if and only if X is a closed subset of $\mathbb{R}^k$. 10

(b) What are separated sets ? Is every pair of disjoint sets separated ? Under what additional hypothesis, a pair of disjoint sets is separated ? 10

4. (a) Show that the set of rational numbers, under usual metric, is not a complete metric space. 8

(b) Prove the nested interval intersection property of reals. 6

(c) State the meaning of functions of bounded variation. Provide three examples of functions, which are not of bounded variation. 6

SECTION—III

5. (a) Prove that countable intersection of dense open subsets of a complete metric space X is dense in X. 8

(b) If f is a differentiable $\mathbb{R} \to \mathbb{R}$ function with $|f'(x)| < 1$, for all $x \in \mathbb{R}$. Does it imply that f has a fixed point in $\mathbb{R}$? 6

(c) Use Heine-Borel property of $\mathbb{R}$ to prove the intermediate value theorem. 6

6. (a) Let f be a continuous bijection from a compact metric space X onto another metric space Y. Prove that f^{-1} is also continuous. 7

(b) Prove that the mapping $x \mapsto \sin\left(\frac{1}{x}\right)$ is continuous on (0, 1), but not uniformly continuous on (0, 1). 7

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function at some $d \in \mathbb{R}$. Prove that both one sided limits $\lim_{x \to d+} f(x)$ and $\lim_{x \to d-} f(x)$ exists. 6

SECTION—IV

7. (a) Prove that monotone real valued function on [0, 1] are Riemann-Stieltjes Integrable on [0, 1]. 6

(b) Let $f : [a, b] \to [0, +\infty)$ be a continuous function such that $\int_0^1 f = 0$. Prove that $f \equiv 0$ on [0, 1]. Is the continuity hypothesis really required ? 8

(c) Let $f : [a, b] \to \mathbb{R}$ such that f^3 is Riemann integrable. Prove that so is f. 6